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ON (p,ω)-PRECISE FUNCTIONS WHOSE DERIVATIVES ARE SINGULAR INTEGRALS

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1. Theorems. As kernels of potentials we shall be concerned only with Riesz and Bessel kernels in $\mathbb{R}^d, d \geq 2$. We write $k_a(x) = |x|^{a-d}$ for $0 < a < d$ and $U_a^f = k_a * f$ for a function f in case the potential is well-defined.

Let α be $(\alpha_1, ..., \alpha_d)$ with integers $\alpha_i \geq 0$, and set $|\alpha| = \alpha_1 + \cdots + \alpha_d$. We shall call α an index of order $|\alpha|$. For a function f we write $D^\alpha f = D_x^\alpha f$ for $\partial^{|\alpha|} f / \partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}$ when this has a meaning. In case $\alpha = (0, ..., 0)$ we write $\alpha = 0$ and let $D^0 f$ mean f.

Let $d \geq 2$, $1 < p < \infty$, Γ be a family of locally rectifiable curves in $\mathbb{R}^d$ and ω a weight. We take the definition of extremal length $\lambda_p(\Gamma; \omega)$ for granted. A function f is called (p,ω)-precise in an open set G if the extremal length of the family of locally rectifiable curves in G along each of which f is not absolutely continuous is infinite and $\int_G |\operatorname{grad} f|^p \omega dx$ is finite.

We announce

Theorem 1. *Let $d \geq 2$, $1 < p < \infty$, ω be a weight in $\mathbb{R}^d$ satisfying Muckenhoupt's A_p condition, $f \in L^{p,\omega}(\mathbb{R}^d)$ and α be an index of order $|\alpha| \geq 0$. Then writing $K^{i,\alpha}$ for $\partial D^\alpha k_{1+|\alpha|} / \partial x_i$, we see that*

$$\lim_{r \to 0} \int_{|x-y|>r} K^{i,\alpha}(x-y) f(y) dy$$

exists in $L^{p,\omega}(\mathbb{R}^d)$. We denote this limit by $T^{i,\alpha} f$. If, in addition, $\int_{\mathbb{R}^d} (1+|x|)^{1-d} |f(x)| dx < \infty$, then $D^\alpha k_{1+|\alpha|} * f$ is (p,ω)-precise, the relation

$$\frac{\partial D^\alpha k_{1+|\alpha|} * f}{\partial x_i} = T^{i,\alpha} f$$

holds a.e. in $\mathbb{R}^d$ for each i, and

$$\| \operatorname{grad}(D^\alpha k_{1+|\alpha|} * f) \|_{p,\omega} \leq \text{const.} \|f\|_{p,\omega}$$

is valid.

Next for $a > 0$ we consider the Bessel kernel

$$G_a(r) = \frac{1}{(4\pi)^{a/2}\Gamma(a/2)} \int_0^\infty e^{-\pi|x|^2/t} e^{-t/(4\pi)} t^{(a-d)/2} \frac{dt}{t}.$$

Theorem 2. *Let d, p, ω, f, α be the same as in Theorem 1. Writing $K_{i,\alpha}$ for $\partial D^\alpha G_{1+|\alpha|} / \partial x_i$, we see that*

$$\lim_{r\to 0} \int_{|x-y|>r} K_{i,\alpha}(x-y) f(y) dy$$

exists in $L^{p,\omega}(\mathbb{R}^d)$. Denote this limit by $T_{i,\alpha} f$. Then $D^\alpha G_{1+|\alpha|} * f$ is (p,ω)-precise, the relation

$$\frac{\partial D^\alpha G_{1+|\alpha|} * f}{\partial x_i} = T_{i,\alpha} f + a_i f$$

holds with some constant a_i a.e. in $\mathbb{R}^d$ for each i, and

$$\|\operatorname{grad}(D^\alpha G_{1+|\alpha|} * f)\|_{p,\omega} \le \text{const.}\, \|f\|_{p,\omega}$$

is valid.

2. Proof. We shall prove only Theorem 1 in this paper. We begin with

Lemma 1. *Let d, p, ω and f be as in Theorem 1. Let Φ_n be an integrable function in $\mathbb{R}^d$ such that $|\Phi_n(x)| \le c_1 n^d$ for $|x| < 1/n$ and $|\Phi_n(x)| \le c_2 n^{-1}|x|^{-d-1}$ for $|x| > 1/n$, where c_1 and c_2 are constants. Set $a_n = \int_{\mathbb{R}^d} \Phi_n(x) dx$ and $h_n = f * \Phi_n$. Then $\|h_n - a_n f\|_{p,\omega} \to 0$ as $n \to \infty$.*

Proof. By a famous Muckenhoupt theorem [Mu, p.222, Theorem 9] $\|Mf\|_{p,\omega} \le$ const. $\|f\|_{p,\omega}$, where Mf is the Hardy-Littlewood maximal function. It follows that $Mf(x) < \infty$ a.e. in $\mathbb{R}^d$. Let us see that f is locally integrable in $\mathbb{R}^d$. In fact, for any compact set K in $\mathbb{R}^d$ we have

$$\int_K |f| dx \le \left(\int_K |f|^p \omega dx\right)^{1/p} \left(\int_K \omega^{1/(1-p)} dx\right)^{1/p'} \le \|f\|_{p,\omega} \left(\int_K \omega^{1/(1-p)} dx\right)^{1/p'} < \infty$$

because $\omega \in A_p$ shows that the last integral is finite.

Now set

$$k(r) = \int_{|y|<r} |f(x-y) - f(x)| dy.$$

Then $r^{-d}k(r) \leq c_3(Mf(x)+|f(x)|) < \infty$ for a.e. $x \in \mathbb{R}^d$. Let x be such a point. Moreover, since f is locally integrable, $r^{-d}k(r) \to 0$ as $r \to 0$ also for a.e. x by a well-known result in integration theory; see, for instance, [R, p.168, Theorem 8.8]. We suppose x has such a property too. Given $\varepsilon > 0$ choose $r_0 > 0$ so that $r^{-d}k(r) < \varepsilon$ if $0 < r < r_0$. In order to show $\lim_{n\to\infty}(h_n - a_n f) = 0$, it is sufficient to consider n such that $1/n < r_0$. We write

$$\begin{aligned} h_n(x) - a_n f(x) &= \left(\int_{|y|<1/n} + \int_{1/n<|y|<r_0} + \int_{|y|>r_0}\right)(f(x-y)-f(x))\Phi_n(y)dy \\ &= I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

Since $|\Phi_n(y)| \leq c_1/n^{-d}$ if $|y| < 1/n$,

$$|I_1(x)| \leq \frac{c_1}{n^{-d}} \int_{|y|<1/n} |f(x-y)-f(x)|dy \to 0$$

as $n \to \infty$ at our x. We note that

$$|I_2(x)| \leq \frac{c_2}{n}\int_{1/n}^{r_0} \frac{1}{r^{d+1}}dk(r) \leq \frac{c_2}{n}\left(\frac{k(r_0)}{r_0^{d+1}} + \varepsilon\int_{1/n}^{r_0}\frac{1}{r^2}dr\right) \leq \frac{c_4}{n}(1+\varepsilon n) \leq 2c_4\varepsilon$$

if n is large. To avoid a repetition of similar computations we shall give a preliminary evaluation before evaluating $I_3(x)$. If $1/n \leq \alpha < \infty$, then

$$\begin{aligned} &\int_\alpha^\infty |(f(x-y)-f(x))\Phi_n(y)|dy \leq \frac{c_2}{n}\int_\alpha^\infty \frac{1}{r^{d+1}}dk(r) \\ &\quad= \frac{c_2}{n}\left(\frac{k(r)}{r^{d+1}}\bigg|_\alpha^\infty + (d+1)\int_\alpha^\infty \frac{k(r)}{r^{d+2}}dr\right) \\ (1)\qquad &\quad\leq \frac{c_2c_3}{n}\lim_{r\to\infty}\frac{Mf(x)+|f(x)|}{r} + (d+1)c_2c_3\frac{Mf(x)+|f(x)|}{n}\int_\alpha^\infty \frac{1}{r^2}dr \\ &\quad= (d+1)c_2c_3\frac{Mf(x)+|f(x)|}{\alpha n}. \end{aligned}$$

By this evaluation we see that $|I_3(x)| \leq (d+1)c_2c_3(Mf(x)+|f(x)|)/(r_0 n) \to 0$ as $n \to \infty$. Accordingly $\limsup_{n\to\infty}|h_n(x) - a_n f(x)| \leq \text{const.}\,\varepsilon$ so that $\lim_{n\to\infty}(h_n(x) - a_n f(x)) = 0$.

Next we shall show that $|h_n - a_n f|$ is dominated by a function, which is independent of n and belongs to $L^{p,\omega}(\mathbb{R}^d)$. We write

$$h_n(x) - a_n f(x) = \left(\int_{|y|<1/n} + \int_{1/n<|y|}\right)(f(x-y)-f(x))\Phi_n(y)dy = I_1(x) + I_2'(x).$$

We observe that $|I_1(x)| \le c_3(Mf(x) + |f(x)|)$, and that $|I_2'(x)|$ is dominated by $(d+1)c_2c_3(Mf(x) + |f(x)|)$ by (1). We recall that $Mf + |f|$ belongs to $L^{p,\omega}(\mathbb{R}^d)$ in virtue of the Muckenhoupt theorem. Consequently we can apply Lebesgue's dominated convergence theorem and obtain

$$\lim_{n\to\infty} \|h_n - a_n f\|_{p,\omega} = \|\lim_{n\to\infty}(h_n - a_n f)\|_{p,\omega} = 0.$$

Thus the lemma is completely proved.

We shall give three more lemmas. Their proofs require a number of properties of (p,ω)-precise functions and so they are omitted. We only refer to the corresponding results in [O].

We shall call a set Ein $\mathbb{R}^d$ (p,ω)-exc. if $\lambda_p(\Gamma;\omega) = \infty$ for the family Γ of curves terminating at the points of E. This can be characterized as a kind of set of capacity zero. We shall say that a property holds (p,ω)-a.e. if the exceptional set is a (p,ω)-exc. set.

Lemma 2. [O, Theorems 4.4.4 and 4.4.5]. *Let $\omega \in A_p$. Let $f_1, f_2, \dots$ be (p,ω)-precise in $\mathbb{R}^d$ and assume*

$$\lim_{n,m\to\infty} \|\operatorname{grad} f_n - \operatorname{grad} f_m\|_{p,\omega} = 0.$$

Then there exist a (p,ω)-precise function f in $\mathbb{R}^d$, a subsequence $\{n_j\}$ and a sequence $\{c_j\}$ of constans such that $\|\operatorname{grad} f_n - \operatorname{grad} f\|_{p,\omega} \to 0$ and $f_{n_j} - c_j \to f$ (p,ω)-a.e. in $\mathbb{R}^d$.

Lemma 3. [O, Corollary to Theorem 4.4.6]. *If a sequence $\{g_n\}$ of (p,ω)-precise functions converges pointwise to a function g (p,ω)-a.e. in $\mathbb{R}^d$ and $\{\operatorname{grad} g_n\}$ is a Cauchy sequence in $L^{p,\omega}(\mathbb{R}^d)$, then g is (p,ω)-precise and $\|\operatorname{grad} g_n - \operatorname{grad} g\|_{p,\omega} \to 0$ as $n \to \infty$.*

Lemma 4. [O, Theorem 4.2.5]. *Let f, g be (p,ω)-precise functions in $\mathbb{R}^d$ which are equal (p,ω)-a.e. in $\mathbb{R}^d$. Then $\operatorname{grad} f = \operatorname{grad} g$ a.e. in $\mathbb{R}^d$.*

Proof of Theorem 1. The first assertion of the theorem is a consequence of Theorem III due to Coifman and Fefferman [CF, Studia Math., 1974]. Assume $\int_{\mathbb{R}^d}(1+|x|)^{1-d}|f(x)|dx < \infty$. This is a necessary and sufficient condition for $U_1^{|f|}$ to be finite a.e. in $\mathbb{R}^d$.

Set $\varphi_n = D^\alpha k_{1+|\alpha|,1/n}$ and $g_n = \varphi_n * f$, where $k_{a,b}(x) = (|x|^2+b^2)^{(a-d)/2}$ in general. We can observe easily

$$\left|\frac{\partial}{\partial x_i}\varphi_n(x-y)f(y)\right| \leq \begin{cases} \text{const.}\, n^d|f(y)| & \text{if } |y| < 2(|x|+1), \\ \text{const.}\, \dfrac{|f(y)|}{|y|^d} & \text{if } |y| \geq 2(|x|+1). \end{cases}$$

We infer that $\partial g_n/\partial x_i = (\partial \varphi_n/\partial x_i) * f$ and it is continuous in $\mathbb{R}^d$. Therefore g_n is absolutely continuous along all locally rectifiable curves in $\mathbb{R}^d$. Setting $K_{1/n}^{i,\alpha}(x) = K^{i,\alpha}\chi_{|\cdot|>1/n}(x)$, we know that $K_{1/n}^{i,\alpha} * f \to T^{i,\alpha}f$ in $L^{p,\omega}(\mathbb{R}^d)$ as $n\to\infty$. We can write

$$\frac{\partial g_n}{\partial x_i}(x) - K_{1/n}^{i,\alpha} * f(x) = \Phi_n * f,$$

where

$$\Phi_n(x) = \begin{cases} \dfrac{\partial}{\partial x_i} D^\alpha\left(|x|^2+\dfrac{1}{n^2}\right)^{(1+|\alpha|-d)/2} & \text{on } |x| < 1/n, \\ \dfrac{\partial}{\partial x_i} D^\alpha\left(|x|^2+\dfrac{1}{n^2}\right)^{(1+|\alpha|-d)/2} - \dfrac{\partial}{\partial x_i} D^\alpha |x|^{1+|\alpha|-d} & \text{on } |x| \geq 1/n. \end{cases}$$

By the mean value theorem we have $|\Phi_n(x)| \leq \text{const.}\, 1/(n^2|x|^{d+1})$ on $|x| \geq 1/n$. Hence $\int_{\mathbb{R}^d}|\Phi_n(x)|dx < \infty$. As in [Mi, p.219, Lemma 4.1] we see that $\int_{\mathbb{R}^d}\Phi_n(x)dx$ vanishes.

In view of Lemma 1 and the equality $\int_{\mathbb{R}^d}\Phi_n(x)dx = 0$, we obtain

$$\text{(1)} \qquad \lim_{n\to\infty}\left\|\frac{\partial g_n}{\partial x_i} - K_{1/n}^{i,\alpha} * f\right\|_{p,\omega} = \lim_{n\to\infty}\left\|\int_{\mathbb{R}^d} f(x-y)\Phi_n(y)dy\right\|_{p,\omega} = 0.$$

We recall that $K_{1/n}^{i,\alpha} * f \to T^{i,\alpha}f$ in $L^{p,\omega}(\mathbb{R}^d)$ as $n\to\infty$. So naturally each $\|K_{1/n}^{i,\alpha} * f\|_{p,\omega}$ is finite. Hence (1) gives $\|\partial g_n/\partial x_i\|_{p,\omega} < \infty$ for each n. The absolute continuity along all locally rectifiable curves being known, it follows that g_n is (p,ω)-precise. From (1) and the fact $\lim_{n\to\infty}\|K_{1/n}^{i,\alpha} * f - T^{i,\alpha}f\|_{p,\omega} = 0$ we infer that $\{\partial g_n/\partial x_i\}$ form a Cauchy sequence in $L^{p,\omega}(\mathbb{R}^d)$. Using Lemma 2 we find a (p,ω)-precise fuction g_0, a subsequence $\{n_j\}$ and a sequence $\{c_j\}$ of constants such that $\|\operatorname{grad}(g_n - g_0)\|_{p,\omega} \to 0$ and $g_{n_j} - c_j \to g_0$ (p,ω)-a.e. The assumption $\int_{\mathbb{R}^d}(1+|x|)^{1-d}|f(x)|dx < \infty$ implies that

$$|g_{n_j}(x)| = |D^\alpha k_{1+|\alpha|,1/n} * f(x)| \leq \text{const.}\, U_1^{|f|}(x) < \infty$$

for a.e. x. Hence we may assume that all c_j are zero so that $g_{n_j} \to g_0$ (p,ω)-a.e.

From (1) it follows that there exists a subsequence of $\{n_j\}$, which will be denoted again by $\{n_j\}$, such that

$$\lim_{j\to\infty}\left(\frac{\partial g_{n_j}}{\partial x_i} - K^{i,\alpha}_{1/n_j} * f\right) = 0 \tag{2}$$

a.e. in $\mathbb{R}^d$. The relations $\lim_{n\to\infty} \|\operatorname{grad}(g_n - g_0)\|_{p,\omega} = 0$ and $\lim_{n\to\infty} \|K^{i,\alpha}_{1/n} * f - T^{i,\alpha} f\|_{p,\omega} = 0$ show that we may assume that $\lim_{j\to\infty} \partial g_{n_j}/\partial x_i = \partial g_0/\partial x_i$ and $\lim_{j\to\infty} K^{i,\alpha}_{1/n_j} * f = T^{i,\alpha} f$ a.e. in $\mathbb{R}^d$ for each i. Taking into account (2) we obtain the equality

$$\frac{\partial g_0}{\partial x_i} = T^{i,\alpha} f \qquad \text{a.e. in } \mathbb{R}^d \text{ for each } i. \tag{3}$$

In the special case $\alpha = 0$ and $f \geq 0$ g_n increases to $k_1 * f = U^f_1$ everywhere in $\mathbb{R}^d$ and $\{\operatorname{grad} g_n\}$ form a Cauchy sequence. Lemma 3 shows that U^f_1 is (p,ω)-precise. In the general case $|D^\alpha k_{1+|\alpha|,1/n}| \leq \text{const.}\, k_1$ and $U^{|f|}_1$ is finite a.e. in $\mathbb{R}^d$. Hence applying Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} D^\alpha k_{1+|\alpha|,1/n} * f(x) = D^\alpha k_{1+|\alpha|} * f(x)$$

at every point x with finite $U^{|f|}_1(x)$. Again by Lemma 3 we infer that $D^\alpha k_{1+|\alpha|} * f$ is (p,ω)-precise.

Next, we recall that $g_{n_j} \to g_0$ as $j \to \infty$ (p,ω)-a.e. and obtain $g_0 = D^\alpha k_{1+|\alpha|} * f$ (p,ω)-a.e. in $\mathbb{R}^d$. Using Lemma 4 and (3) we derive

$$\frac{\partial D^\alpha k_{1+|\alpha|} * f}{\partial x_i} = \frac{\partial g_0}{\partial x_i} = T^{i,\alpha} f$$

a.e. in $\mathbb{R}^d$ for each i. Finally since $\|T^{i,\alpha} f\|_{p,\omega} \leq \text{const.}\, \|f\|_{p,\omega}$, $\|\operatorname{grad}(D^\alpha k_{1+|\alpha|} * f)\|_{p,\omega} \leq \text{const.}\, \|f\|_{p,\omega}$ as announced.

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